3.3

A Fair Minimax Theorem for Two-Person (Zero-Sum) Games Involving Finitely Additive Strategies

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ABSTRACT

In this chapter we discuss the sensitivity of the minimax theorem to the cardinality of the set of pure strategies. In this light, we examine an infinite game due to Wald and its solutions in the space of finitely additive (f.a.) strategies.

Finitely additive joint distributions depend, in general, upon the order in which expectations are composed out of the players' separate strategies. This is connected to the phenomenon of "non-conglomerability" (so-called by deFinetti), which we illustrate and explain. It is shown that the player with the "inside integral" in a joint f.a. distribution has the advantage.

In reaction to this asymmetry, we propose a family of (weighted) symmetrized joint distributions and show that this approach permits "fair" solutions to fully symmetric games, e.g., Wald's game. We develop a minimax theorem for this family of symmetrized joint distributions using a condition formulated in terms of a pseudo-metric on the space of f.a. strategies. Moreover, the resulting game

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can be solved in the metric completion of this space. The metrical approach to a minimax theorem is contrasted with the more familiar appeal to compactifications, and we explain why the latter appears not to work for our purposes of making symmetric games "fair." We conclude with a brief discussion of three open questions relating to our proposal for f.a. game theory.

INTRODUCTION

In this essay we derive results for finitely additive (mixed) strategies in two-person, zero-sum games with bounded payoffs. We establish a minimax theorem which is novel in that it allows for joint (finitely additive) distributions which make symmetric (bounded) games fair. That is, the minimax value of a fully symmetric game is 0 under our proposal.

In section I we review the sensitivity of the familiar minimax theorem (of von Neumann and Morgenstern, 1947) to the cardinality of the set of pure strategies. That result, which uses mixed strategies taken from the class of countably additive probabilities, does not apply when the set of pure strategies is infinite. A simple game due to Wald (1950), "Pick the Bigger Integer" (Example 1.1), illustrates the problem. (In this game, the payoff is 0 if both players pick the same integer, otherwise the winner receives 1.) When all strategies are countably additive, this game has no value. If, however, only one player is allowed to use a (merely) finitely additive mixed strategy, Wald's game has a value and that player wins. Allowing both players to use finitely additive mixed strategies leads to a value for the game (as shown by Heath and Sudderth, 1972), but it has the unfortunate consequence that the value depends upon the order of integration over the two mixed strategies. We relate this phenomenon, as it appears in "Pick the Bigger Integer," to P. Lévy's 1930 example of what deFinetti (1972) calls "non-conglomerability" of finitely additive probability.

We find that for all games the player with the "inside" integral occupies the favored position (Theorem 2.2). This means that even for games with symmetric payoffs, as in Wald's game, there may be no symmetry reflected in the value of the game if it is solved by using a particular order of integration. Also in section II we investigate joint finitely additive distributions created from the players' two mixed strategies by taking convex combinations of the two "extreme" joint distributions, where the "extreme" versions of a game correspond to fixing the order of integration first one way and then the other. This leads to a parametrized class of joint distributions, r_w, indexed by a weight w, $0 \leq w \leq 1$. (That is, the "extreme" versions of the game correspond to the values $w = 0$ and $w = 1$.) We prove the minimax theorem (Theorem 2.3) for the set of joint distributions r_w under an assumption (Condition $\mathcal{A}$) expressed in terms of metrical properties of the space of mixed strategies. A simple corollary is that for fully symmetric games (where payoffs satisfy $f(s, t) = -f(t, s)$ and with $w = 0.5$) the game is fair, i.e., the value for the game is 0. Also under condition $\mathcal{A}$, Theorem 2.4 establishes the existence of minimax strategies in the (metric) completion of the space of mixed strategies.

We illustrate the phenomenon that the solution to a game may fail to be a mixture of the minimax solutions from the two "extreme" versions of that game (Example 3). There is an important non-convexity of minimax solutions associated with joint distributions formed by convex combinations of the two "extreme" versions of a game.

In section III we provide a brief account of some related literature. (We defer our review of others' work until section III to allow a contrast with the position taken in this report.) In section IV we indicate a connection between our treatment of games and "improper" priors, and our concluding section V addresses several open questions which we find of interest.

I. THE EFFECT OF INFINITELY MANY PURE STRATEGIES ON THE MINIMAX THEOREM

I.1. *Two-Person, Zero-Sum Games with Infinitely Many Pure Strategies*

In a two-person, zero-sum game, player-1 has pure strategies $s \in S$ and player-2 has pure strategies $t \in T$. Let $f(s, t)$ be the real-valued payoff to player-1, so that $-f(s, t)$ is the payoff to player-2, when player-1 uses strategy s and player-2 uses strategy t. Allow the players to have mixed strategies, i.e., player-1 may use a distribution p ($\in P_\sigma$) on S and player-2 may use a distribution q ($\in Q_\sigma$) on T; for P_σ and Q_σ sets of (σ-additive) probabilities. Then, the (expected) value of strategy pair (p, q) is (assuming f is bounded and measurable with respect to the product measure, $p \times q$):

$$E_{p\times q}f(s,t) - E_p[E_q f(s,t)] = E_q[E_p f(s,t)].$$

That is, the joint distribution is the product measure which, by Fubini's theorem, may be written as the double integral in either order.

Let S and T be finite sets; then (von Neumann and Morgenstern, 1947) the fundamental result of two-person, zero-sum games asserts:

Theorem 1.1 ("Minimax").

$$\sup_{P\sigma} \inf_{Q\sigma} E_{p\times q}f(s,t) = \inf_{Q\sigma} \sup_{P\sigma} E_{p\times q}f(s,t) = V \qquad \text{(i)}$$

That is, *the game has a value V.* Also:

$$\exists p^* \forall q E_{p^*\times q} f(s,t) \geq V \text{ [maximin strategy]} \qquad \text{(ii)}$$

and

$$\exists q^* \forall p E_{p\times q^*} f(s,t) \leq V \text{ [minimax strategy]}$$

Thus, the strategy pair (p^*, q^*) solves the game. □

This minimax result, even the fact that the game has a value, depends upon there being only finitely many pure strategies. Wald (1950) provides an elegant counterexample when the sets S and T are denumerable.

Example 1.1 ("Pick the Bigger Integer"). Let the pure strategies, $S = T = \{1, 2, \ldots,\}$, be the positive integers. Define the payoff function to player-1 (which is the negative of the payoff to player-2) by:

$$\begin{aligned} f(s,t) &= 1, \quad \text{if } s > t \\ &= 0, \quad \text{if } s = t \\ &= -1, \quad \text{if } s < t. \end{aligned}$$

Then,

$$\sup_P \inf_Q E_{p\times q}f(s,t) = -1$$

while

$$\inf_Q \sup_P E_{p\times q}f(s,t) = +1.$$

Proof. For each σ-additive measure μ on the positive integers, given ε $(0 < \varepsilon < 1)$, there is another σ-additive measure v and integer n_ε where $v\{n: n > n_\varepsilon\} > 1 - \varepsilon$ while $\mu\{n{:}n < n_\varepsilon\} > 1 - \varepsilon$. □

Thus, game-1 is without a value in the class of countably additive strategies.

Suppose we allow *one* player the use of a finitely additive (f.a.) mixed strategy in Wald's game. Then we can show that the game is determined and that player wins.

Definition. Call p a *purely finitely additive* (p.f.a.) probability if, given $\varepsilon > 0$, there is a denumerable partition $\pi = \{h_i : i = 1, \ldots\}$ with $\Sigma_{h_i \in \pi} p(h_i) < \varepsilon$. Also, we refer to these as *diffuse* probabilities.

If player-1 adopts a diffuse probability $p_d(n) = 0$ $(n = 1, \ldots)$, but player-2 is restricted to the set Q_σ, then $\inf_{Q_\sigma} E_{p_d \times q} f(s, t) = +1$. Likewise, when player-2 adopts a diffuse probability $q_d(n) = 0$ $(n = 1, \ldots)$, but player-1 is restricted to the set P_σ, then $\sup_{P_\sigma} E_{p \times q_d} f(s, t) = -1$.

This notation is warranted because, on countable spaces, if one of the players uses a countably additive mixed strategy, the order of integration is irrelevant. That is,

Lemma 1.1.

$$\forall(q \in Q_\sigma) E_{p_d}[E_q f(s,t)] = E_q[E_{p_d} f(s,t)] = +1$$

Also,

$$\forall(p \in P_\sigma) E_{q_d}[E_p f(s,t)] = E_p[E_{q_d} f(s,t)] = -1.$$

(The proof is straightforward and is omitted.)

On the other hand, suppose we allow both players in Wald's game to use finitely additive mixed strategies (p, q). The solution now depends upon how the joint strategy, "$p \times q$", is defined. In general, with finitely additive distributions p and q, $E_q[E_p f(s, t)] \neq E_p[E_q f(s, t)]$.

Example 1.2 (attributed to P. Lévy by deFinetti, 1972). Consider a diffuse probability r on the set of all pairs $\langle s, t \rangle$, for s and t positive integers, with the following two restrictions:

$$r(\langle s, t \rangle) = 0,$$

that is, r is 0 on finite sets; and

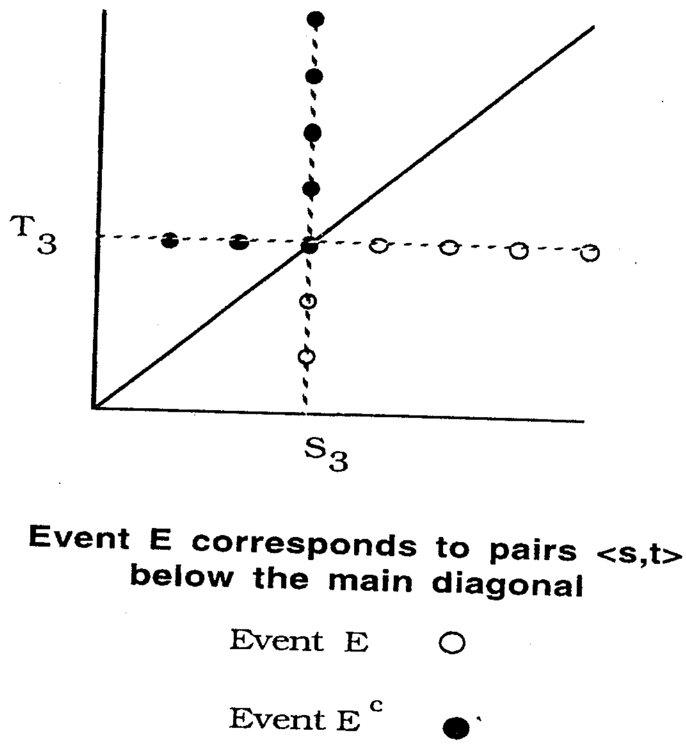

Figure 1. Diagram for P. Lévy's example. Only finitely many points on each vertical section lie below the diagonal. Only finitely many points on each horizontal section lie above the diagonal.

$$r(\langle s,t\rangle | F) = 0 \quad \text{if } F \text{ is an infinite set,}$$

that is, conditionally, r is again diffuse given an infinite set F.

Define the events:

$$E = \{\langle s,t\rangle : s > t],$$

$$S_m = \{\langle s,t\rangle : s = m\} \quad (m = 1, \ldots),$$

and

$$T_n = \{\langle s,t\rangle : t = n\} \quad (n = 1, \ldots).$$

Then, $r(E | S_m) = 0$ for $m = 1, \ldots,$ yet $r(E | T_n) = 1$ for n $= 1, \ldots$. That is, conditioned on each vertical section, the r-probability of an outcome below the main diagonal is 0. However, conditioned on each horizontal section, the r-probability of the same event is 1.

Figure 1 illustrates what is happening.

I.2. *Non-conglomerability of Finitely Additive Probabilities*

Definition (Dubins, 1975). Say that probability p is *conglomerable* in the (denumerable) partition $\pi = \{h_i: i = 1, \ldots\}$ provided that, for each bounded random quantitiy X and $\forall(k_1, k_2)$, if

$$k_1 \leq E(X|h_i) \leq k_2 \quad (i = 1, \ldots)$$

then

$$k_1 \leq E(X) \leq k_2,$$

where $E(\bullet)$ means expectation with respect to p.

Equivalently (Dubins, 1975), p is conglomerable in partition π just in case p is disintegrable in π, and for the special case of an event E (identified with its indicator function)

$$p(E) = \int_{h \in \pi} p(E|h)\,dp(h), \quad \text{for all } E.$$

In Lévy's example, with the two partitions $\pi_1 = \{s_i: i = 1, \ldots\}$ and $\pi_2 = \{t_j: j = 1, \ldots\}$, we see that

$$\int_{s \in \pi_1} p(E|s)\,dp(s) = 0 \quad \text{and} \quad \int_{t \in \pi_2} p(E|t)\,dp(t) = 1.$$

So p fails to be conglomerable in at least one of the two (denumerable) partitions π_1 and π_2.

The lack of conglomerability is endemic to merely finitely additive probabilities. That is, each f.a. probability that is *not* countably additive experiences non-conglomerability in some denumerable partition (Schervish et al., 1984). More precisely, we can say this. Each f.a. probability p has a (unique) decomposition into a convex combination of a σ-additive probability p_σ and a purely finitely additive probability p_d:

$$\forall p \exists! (0 \leq a \leq 1) \quad p = ap_d + (1-a)p_\sigma$$

and p_d [or p_σ] is unique if a [or if $1 - a$] is positive (Yoshida and Hewitt, 1952). The quantity a is the least upper bound on the extent of non-conglomerability of p with respect to events. That is, given a f.a. probability p, for each $\varepsilon > 0$ there is a denumerable partition $\pi = \{h_i: i = 1, \ldots\}$ and event E where,

$$p(E) - p(E|h_i) > a - \varepsilon, \quad \text{for all } i.$$

II.1. *A Proposal for the (Finitely Additive) Joint Distribution "$p \times q$"*

Reconsider Wald's game ("Pick the Bigger Integer"). Observe that in Lévy's example, the event E corresponds to the outcomes where player-1 wins. The non-conglomerability of the probability r (in Lévy's example) illustrates the effect of changing the order of integration in creating a joint distribution on $S \times T$ from the two diffuse "marginal" distributions, p and q. These marginal probabilities correspond to the players' strategies on S and on T (respectively). Let P and Q denote the sets of finitely additive mixed strategies on sets S and T. Thus, when players use (diffuse) purely finitely additive probabilities (p_d, q_d) which assign probability zero to each pure strategy, then

$$\forall (p \in P) E_p[E_{q_d} f(s,t)] = -1 \quad \text{and} \quad \forall (q \in Q) E_q[E_{p_d} f(s,t)] = +1.$$

In particular,

$$E_{p_d}[E_{q_d} f(s,t)] = -1 \neq +1 = E_{q_d}[E_{p_d} f(s,t)].$$

How shall we define the joint distribution that results when player-1 adopts the f.a. strategy p and player-2 adopts strategy q? The condition which motivates our solution, below, is to allow that symmetric bounded games, such as Wald's, admit a solution which makes them fair. That is, when $f(s,t) = -f(t,s)$ for each pure strategy pair (s,t) (and when payoffs are bounded), so the game is symmetric, we require there to be a finitely additive solution to the game that makes it a "draw." We require of such games that they have a value 0.

Proposal. Given f.a. probabilities p and q on S and T (respectively), and given $0 \leq w \leq 1$, adopt the joint probability r_w on the power set $\mathcal{P}[S \times T]$, as follows. For event $E \in \mathcal{P}[S \times T]$,

$$r_w[E] = w\int_T \int_S \chi_E \, dp \, dq + (1-w)\int_S \int_T \chi_E \, dq \, dp,$$

where χ_E is the indicator function for event E. The joint distribution r_w can be thought of as a w-weighted coin-flip between the two joint distributions obtained by fixing the order of integration.

Generally, given p, q and a (bounded) function $f(s,t)$ on $S \times T$, define the E_{r_w}-expectation of f as:

$$E_{r_w}(p,q) = wE_q[E_p f(s,t)] + (1-w)E_p[E_q f(s,t)].$$

The parameter w weights the contribution to the joint expectation E_{r_w} on $S \times T$ of the two "extreme" distributions $E_{r_1} = E_q[E_p f]$ and $E_{r_0} = E_p[E_q f]$. As we explain next (Theorem 2.2), these two are "extreme" in the sense that, in zero-sum games, $w = 1$ favors the first (p-)player, whereas $w = 0$ favors the second (q-)player. Each player prefers the E_{r_w}-game where his expectation has the "inside" integral.

Heath and Sudderth (1972) show that (for games with bounded payoffs) when the joint distribution is determined by fixing the order of integration, then the game has a value. That is, their result is:

Theorem 2.1.

$$V_1 = \sup_P \inf_Q E_q[E_p f(s,t)] = \inf_Q \sup_P E_q[E_p f(s,t)]$$

and

$$V_2 = \sup_P \inf_Q E_p[E_q f(s,t)] = \inf_Q \sup_P E_p[E_q f(s,t)].$$

In our notation, V_1 is the value of the E_{r_1}-game, and V_2 is the value of the E_{r_0}-game. It is a simple corollary to the Heath-Sudderth result that there are minimax strategies which achieve these values.

Corollary 2.1. Corresponding to the E_{r_1}-game there are strategies, (p_1, q_1) such that

$$V_1 = \inf_Q E_{r_1}(p_1, q) = \sup_P E_{r_1}(p, q_1),$$

and corresponding to the E_{r_0}-game there are strategies (p_2, q_2) such that

$$V_2 = \inf_Q E_{r_0}(p_2, q) = \sup_P E_{r_0}(p, q_2). \quad \square$$

This corollary also appears as Theorem 2.1 in Kindler (1983).

Theorem 2.2 ($\forall w$).

$$V_2 \le \sup_P \inf_Q E_{r_w}(p,q) \le \inf_Q \sup_P E_{r_w}(p,q) \le V_1$$

Proof. The players' minimax strategies (each taken from his *dis*-favored game) provide the desired bounds. Corollary 2.1 ensures these strategies exist. Specifically,

$$\sup_P \inf_Q E_{r_w}(p, q) \geq \inf_Q E_{r_w}(p_2, q) \geq w(\inf_Q E_q[E_{p_2} f(s, t)])$$
$$+ (1-w)\inf_Q E_{p_2}[E_q f(s, t)] \geq V_2.$$

Only the third of these inequalities requires an explanation. It follows from the two facts: (i) $\inf_Q E q_2[E_q f(s, t)] = V_2$, since q_2 solves the E_{r_0} game; and (ii) $\inf_Q E_q[E_{p_2} f(s, t)] \geq V_2$, since, if on the contrary ($\inf_Q E_q[E_{p_2} f(s, t)] < V_2$), then $\exists(t^* \in T)$ such that $E_{p_2} f(s, t^*) < V_2$. But that contradicts p_2 as a minimax solution to the E_{r_0}-game. A similar argument, with "p", "q", and inequalities all interchanged, proves the result about V_1. □

Hence, for all bounded games, $V_2 \leq V_1$, and a player's advantage is with the "inside" integral.

To express our minimax theorems concerning the E_{r_w}-game, we pseudo-metrize the set of strategies.

Definition. Say that two strategies for a player are *equivalent in the E_{r_w}-game* provided they have the same (expected) value against each possible strategy of the opponent. (Denote this relation by ≡, where the game is identified by context.)

That is,

$$(p_1 \equiv p_2) \text{ in game } E_{r_w} \text{ if and only if } (\forall q) E_{r_w}(p_1, q) = E_{r_w}(p_2, q);$$

likewise

$$(q_1 \equiv q_2) \text{ in game } E_{r_w} \text{ if and only if } (\forall p) E_{r_w}(p, q_1) = E_{r_w}(p, q_2).$$

It is obvious that ≡ is an equivalence relation.

Consider the pseudo-metrics ρ_P *on* P-strategies and ρ_Q on Q-strategies defined by

$$\rho_P(p_1, p_2) = \sup_Q |E_{r_w}(p_1, q) - E_{r_w}(p_2, q)|$$

and

$$\rho_Q(q_1, q_2) = \sup_P |E_{r_w}(p, q_1) - E_{r_w}(p, q_2)|.$$

Lemma 2.1. $\rho_P(\rho_Q)$ is a pseudo-metric on the set P (set Q). □

Wald (1950, chapter 2) introduced the same pseudo-metrics to deal with countably additive mixed strategies. The proof that they are pseudo-metrics is the same as in the countably additive case.

Clearly, $\rho_P(p_1, p_2) = 0$ just in case $(p_1 \equiv p_2)$, and similarly with ρ_Q; thus, these two pseudo-metrics are metrics on their respective ≡-equivalence classes of strategies. The pseudo-metrics express by how much two strategies may be separated through the choice of the opponent's reply.

Next, we formulate a sufficient condition for existence of a value for the E_{r_w}-game. The condition concerns approximately maximin strategies for one player and corresponding approximately best responses by the other player. Some notation is needed, first, to make these concepts precise.

Suppose $\sup_P \inf_Q E_{r_w}(p, q) = a$. Then, given $\varepsilon > 0$, $\forall p \in P, \exists q \in Q$ with $E_{r_w}(p, q) \leq a + \varepsilon$. Given p, define the quantity:

$$v(p) = \inf_Q E_{r_w}(p, q)$$

and the set

$$R^{\varepsilon}(p) = \{q: E_{r_w}(p, q) \leq v(p) + \varepsilon\}.$$

$v(p)$ is the (limit of the) *value* of best replies q-player can make against strategy p. $R^{\varepsilon}(p)$ is the *set* of ε-best replies q-player has against the strategy p.

Let

$$B_{\rho_Q}(q^*, \varepsilon) = \{q: \rho_Q(q, q^*) \leq \varepsilon\}, \text{ the set of } q\text{'s near to } q^*.$$

Also, because $\sup_P \inf_Q E_{r_w}(p, q) = a$, given $\varepsilon > 0, \exists p \in P, \forall q \in Q$ such $E_{r_w}(p, q) \geq a - \varepsilon$.

Define the set

$$P^{\varepsilon} = \{p: \forall q E_{r_w}(p, q) \geq a - \varepsilon\}.$$

P^{ε} is the set of ε-maximin strategies for p-player. Clearly, the sets $R^{\varepsilon}(p)$ and P^{ε} are nonempty and convex. Next, we state the condition under which we prove our minimax theorems. Observe that it is formulated asymmetrically between the two players. We discuss this and other features of the condition below.

Condition $\mathcal{A}$.

$$\exists(k > 0), \forall(\delta > 0), \exists(0 < \varepsilon \leq \delta) \exists p \in P^{\varepsilon} \exists(n > 0; q_i \in Q, 1 \leq i \leq n)$$
$$\text{with } R^{2\varepsilon}(p) \subseteq \cup_i B_{\rho_Q}(q_i, k\varepsilon).$$

Condition $\mathcal{A}$ requires that, for each small $\varepsilon > 0$, there is some ε-maximin strategy p, each of whose 2ε-best replies is $k\varepsilon$-near to (in the sense of ρ_Q) one of some finitely many q-strategies. In simpler words, there is a "safe" p-strategy $[p \in P^{\varepsilon}]$ whose set of "best" responses $[R^{2\varepsilon}(p)]$ is covered by a finite number $[n]$ of "small" balls $[B_{\rho_Q}(q_i, k\varepsilon)]$. Within ε-approximations, Condition $\mathcal{A}$ is that there exists some maximin p-strategy, where each good q-reply to p is close to one of some finite collection of q's.

Condition $\mathcal{A}$ is truly asymmetric; it may be satisfied under one order of the players' strategies, but not with roles reversed. For example, consider an extreme version of Wald's game, Example 1.1, with $w = 1$. Recall, $E_{r_1}(p, q) = E_q E_p[f(s, t)]$. The p-player has the advantage (Theorem 2.2) and the game's value is 1. According to the (corollary to the) Heath-Sudderth theorem (Theorem 2.1), minimax strategies exist. For p-player, the minimax strategies all belong to the same ρ_P-equivalence class – any diffuse strategy, p_d, is minimax and only diffuse strategies are minimax for the first player. However, p_d is an "equalizer" strategy: $\forall_q$, $E_{r_1}(p_d, q) = 1$. Hence, all of Q is the set of "best" responses to p_d.

Since $\rho_Q(q, q') \geq 1$ whenever q and q' are different (point-mass) pure strategies, there is no $p \in P^{\varepsilon}$ satisfying Condition $\mathcal{A}$. The "best" responses to p_d are not contained within finitely many small ρ_Q-neighborhoods. Nonetheless, by considering Condition $\mathcal{A}$ with the players' roles reversed, we discover that, against a diffuse strategy q_d, all good p-responses in the E_{r_1}-game are near to the equivalence class of diffuse strategies, represented by the strategy p_d. That is, with the alternative reading, Condition $\mathcal{A}$ is satisfied using a single ρ_P-neighborhood of p-strategies near to $p_d (k = 2)$.

It is easy to verify that Condition $\mathcal{A}$ obtains in the non-extreme versions of Wald's game: all diffuse strategies (p_d or q_d) lie in the same equivalence class for that strategy space and, in fact, each such strategy is minimax. For $0 < w < 1$ and for sufficiently small ε, all good responses to p_d are close (in the sense of ρ_Q) to q_d. Thus, there is a single neighborhood ($n = 1$, $k = 2$) which satisfies Condition $\mathcal{A}$ and it applies to either player. For non-extreme versions of Wald's game, $\mathcal{A}$ obtains both ways.

Theorem 2.3. Provided Condition $\mathcal{A}$ holds,

$$\sup_P \inf_Q E_{r_w}(p, q) = \inf_Q \sup_P E_{r_w}(p, q).$$

Proof. Clearly, $\sup_P \inf_Q E_{r_w}(p, q) \le \inf_Q \sup_P E_{r_w}(p, q)$, regardless of Condition $\mathcal{A}$. We argue for the reverse inequality using two lemmas (which do not require Condition $\mathcal{A}$).

Lemma 2.2. Let $\sup_P \mathit{inf}_Q E_{r_w}(p, q) = a$. Then there is a finitely additive measure, μ, on the space of Q-strategies with the property that $\forall p \in P, \int_Q E_{r_w}(p, q)\, d\mu \le a$. □

Proof (of lemma). This proof is very similar to that of Heath and Sudderth's (1972, p. 2072) "Theorem 1." Consider the set $\mathcal{B}$ of all bounded functions from Q to the reals, $\Re$. For each $p \in P$, $E_{r_w}(p, \bullet) \in \mathcal{B}$. Let $K_1 = \{f \in \mathcal{B}: \forall q, f(q) > a\}$. Let $K_2 = \{E_{r_w}(p, \bullet): p \in P\}$. The assumption, $\sup_P \inf_Q E_{r_w}(p, q) = a$, implies $K_1 \cap K_2 = \phi$. Clearly, K_1 and K_2 are convex. By Theorem 8 (p. 417 of Dunford and Schwartz, 1958), there exist a scalar c and a non-zero linear function π on $\mathcal{B}$ with the following two properties:

$$\forall p \in P, \forall h \in K_1, \quad \pi[E_{r_w}(p, \bullet)] \le c \le \pi(h);$$

$$\text{If } \lim_{n \to \infty} h_n = h \text{ uniformly on } Q, \text{ then } \lim_{n \to \infty} \pi(h_n) = \pi(h).$$

Since π is non-zero and linear and since each constant function greater than a belongs to K_1, it follows that $\pi(1) > 0$. Normalize π so that $\pi(1) = 1$. Then it is clear that $c \le a$ since $\pi(a) = a$ and the constant functions $a = \varepsilon \in K_1$ (whenever $\varepsilon > 0$). It follows that $\pi[E_{r_w}(p, \bullet)] \le a$.

The proof is concluded by showing there exists a finitely additive probability μ on Q such that, $\forall h \in \mathcal{B}, \pi(h) = \int_Q h(q) d\mu(q)$. Consider $C \in 2^Q$, a set of q's. Given $\varepsilon > 0$, let $h_\varepsilon(q) = a + \varepsilon + \chi_C(q)$. *So* $h_\varepsilon \in K_1$ and $\pi(\chi_C) \ge 0$ for each C. Define $\mu(C) = \pi(\chi_C)$. Since π is linear, μ is finitely additive; and as $\pi(1) = 1$, μ is a probability. If h is a simple function, by linearity of π, then $\pi(h) = \int_Q h(q) d\mu(q)$. Because every bounded function can be uniformly approximated by simple functions, we obtain the desired representation: $\forall h \in \mathcal{B}, \pi(h) = \int_Q h(q) d\mu(q)$. □

Based on Lemma 2.2, define the function $\vartheta(p) = \int_Q E_{r_w}(p, q) d\mu$ and denote by $I^\varepsilon(p)$ the set $I^\varepsilon(p) = \{q: |E_{r_w}(p, q) - \vartheta(p)| \le \varepsilon\}$.

That is, $I^\varepsilon(p)$ is the set of q-strategies whose value against p is close to the integral $\vartheta(p)$.

Lemma 2.3. The family $\{I^\varepsilon(p): p \in P, \varepsilon > 0\}$ has the finite intersection property. □

Proof (of lemma). We give the proof in two cases, depending on the size, n, of the finite family $\{I^{\varepsilon_j}(p_j): p_j \in P, \varepsilon_j > 0, j = 1, \ldots, n\}$. First we argue for the elementary case, $n = 1$, which we generalize to the other case, $n > 1$.

Define $L = \inf_{(S\times T)} f(s, t)$, $U = \sup_{(S\times T)} f(s, t)$, and $d = U - L$. Let $N \geq 2d/\varepsilon$. For $i = 1, \ldots, N - 1$, define $g_i(p) = \{q: L + d(i-1)/N \leq E_{r_w}(p, q) < L + di/N\}$, and for $i = N$, let the last inequality be $\leq$. Let $c_{i,p} = \mu(g_i(p))$, where μ is the f.a. measure taken from Lemma 2.2. For each i where $g_i(p) \neq \varnothing$, let $q_i \in g_i(p)$ (for other i, let q_i be arbitrary). Define $q_{\varepsilon,p} = \Sigma_i \overset{N}{=}1\ c_{i,p}q_i$. It is evident that $E_{r_w}(p, q_{\varepsilon,p}) = \Sigma_i \overset{N}{=}1\ c_{i,p}E_{r_w}(p, q_i)$. We conclude the case ($n = 1$) by showing that $q_{\varepsilon,p} \in I^{\varepsilon}(p)$.

Define the simple function $h_{p,\varepsilon}: Q \to \Re$ as: $h_{p,\varepsilon}(q) = E_{r_w}(p, q_i)$, for $q \in g_i(p)$. It is easy to see that $\forall q$, $|h_{p,\varepsilon}(q) - E_{r_w}(p, q)| < \varepsilon$. Hence, $|\int_Q h_{p,\varepsilon}(q)d\mu - \vartheta(p)| < \varepsilon$. Since $h_{p,\varepsilon}(q)$ is a simple function, constant on all $q \in g_i(p)$, we have that $\int_Q h_{p,e}(q)d\mu = \Sigma_i \overset{N}{=}1\ c_{i,p}E_{r_w}(p, q_i)$, which proves the point.

For $n > 1$, we simplify by noting that $I^{\varepsilon'}(p) \supseteq I^{\varepsilon}(p)$ whenever $\varepsilon' \geq \varepsilon$. That is, without loss of generality we take $\varepsilon = \min\{\varepsilon_i\}$ and prove finite intersection property for $\{I^{\varepsilon}(p_j)\}$. Consider the common refinement of the partitions generated by the Nn sets, $g_i(p_j)$. For each $(i_1, \ldots, i_n) \in \{1, \ldots, N\}^n$, let

$$g_{i_1,\ldots,i_n} = \bigcap_{j=1}^{n} g_{i_j}(p_j)$$

As before, let $q_{i_1,\ldots,i_n} \in g_{i_1,\ldots,i_n}$, when the latter is non-empty; otherwise, let $q_{i_1,\ldots,i_n}$ be arbitrary. Let $c_{i_1,\ldots,i_n} = \mu(g_{i_1,\ldots,i_n})$ and let Σ'_j denote the $n - 1$ fold summation over all the indices other than the jth index. Set $c^j_{i_j} = \Sigma'_j c_{i_1,\ldots,i_n}$. Last, define $q^j_{i_j} = (1/c^j_{i_j})\Sigma'_j c_{i_1,\ldots,i_n} q_{i_1,\ldots,i_n}$. Since $q_{i_1,\ldots,i_n} \in g_{i_j}(p_j)$ for each j and i_j, it follows from the convexity of $g_{i_j}(p_j)$ that $q^j_{i_j} \in g_{i_j}(p_j)$. Therefore, select

$$q^* = \sum_{i_j=1}^{N} c^j_{i_j} q^j_{i_j} = \sum_{i_1=1}^{N} \cdots \sum_{i_n=1}^{N} c_{i_1,\ldots,i_n} q_{i_1,\ldots,i_n}$$

We then have that $q^* \in I^{\varepsilon}(p_j)$, for $j = 1, \ldots, n$. □

Thus, Lemma 2.3 asserts that the intersection of finitely many $I^{\varepsilon}(p)$ sets is not empty,

$$\varnothing \neq \bigcap_j I^{\varepsilon_j}(p_j) \quad (j = 1, \ldots, m).$$

Lemma 2.4. Given a p-strategy which is ε-maximin, $p \in P^{\varepsilon}$, then $R^{2\varepsilon}(p) \supset I^{\varepsilon}(p)$. □

Proof (of lemma). To see that $R^{2\varepsilon}(p) \supseteq I^{\varepsilon}(p)$ when $p \in P^{\varepsilon}$, note that $\forall\, q \in I^{\varepsilon}(p)$, $a - \varepsilon \leq E_{r_w}(p, q) \leq a + \varepsilon$. The first inequality arises from the fact that, since $p \in P^{\varepsilon}$, $\forall q \in Q$, $E_{r_w}(p, q) \geq a - \varepsilon$. The second inequality obtains because, $\forall p$, $\vartheta(p) \leq a$; hence, $\forall q \in I^{\varepsilon}(p)$, $E_{r_w}(p, q) \leq a + \varepsilon$. Last, observe that for $p \in P^{\varepsilon}$, $R^{2\varepsilon}(p) \supseteq \{q{:}a - \varepsilon \leq E_{r_w}(p, q) \leq a + \varepsilon\}$. □

To complete the theorem we argue indirectly that $\sup_P \inf_Q E_{r_w}(p, q) \geq \inf_Q \sup_P E_{r_w}(p, q)$. Assume (on the contrary) that $\inf_Q \sup_P E_{r_w}(p, q) = b > a$. Choose $0 < \varepsilon < (b - a)/(k + 2)$. By Condition $\mathcal{A}$, there is a p-strategy in P^{ε}, denoted by p^*, with the property that $R^{2\varepsilon}(p^*) \subseteq \cup_i B\rho_Q(q_i, k\varepsilon)$, for some finite number of q's, $(i = 1, \ldots, n)$. The assumption that $\inf_Q \sup_P E_{r_w}(p, q) = b$, entails that for each q-strategy there is a p-strategy where $E_{r_w}(p, q) \geq b - \varepsilon$. Hence, for each q_i $(i = 1, \ldots, n)$ there is a p_i where $E_{r_w}(p_i, q_i) \geq b - \varepsilon$. But then if $q \in B\rho_Q(q_i, k\varepsilon)$, $q \notin I^{\varepsilon}(p_i)$. In other words, $B\rho_Q(q_i, k\varepsilon) \cap I^{\varepsilon}(p_i) = \varnothing$. (This follows because $\forall q \in I^{\varepsilon}(p_i)$, $E_{r_w}(p_i, q) \leq a + \varepsilon$; however, $\forall q \in B\rho_Q(q_i, k\varepsilon)$, $E_{r_w}(p_i, q) > b - (k+1)\varepsilon$.) Thus, $R^{2\varepsilon}(p^*) \cap [\cap_i I^{\varepsilon}(p_i)] = \varnothing$. According to Lemma 2.4, then $I^{\varepsilon}(p^*) \cap [\cap_i I^{\varepsilon}(p_i)] = \varnothing$, which contradicts Lemma 2.3.

In light of Theorem 2.3, denote by V_w the value of the game. It is a simple corollary that, provided Condition $\mathcal{A}$ obtains, symmetric games are fair, i.e., for symmetric games $V_{0.5} = 0$.

Corollary 2.2. Under the condition for Theorem 2.3, if $S = T$ and $f(s, t) = -f(t, s)$, so the game is symmetric, then $V_{0.5} = 0$, i.e., then the $E_{r_{0.5}}$-game is fair.

Proof. By the symmetry of the game, and since the two players have identical strategy spaces $(P = Q)$, observe that for each pair (p, q) $E_{r_{0.5}}(p, q) = -E_{r_{0.5}}(q, p)$. We argue indirectly. Suppose $\sup_P \inf_Q E_{r_w}(p, q) = \inf_Q \sup_P E_{r_w}(p, q) = b > 0$. Then there exists a p' strategy such that for all q strategies, $E_{r_{0.5}}(p', q) \geq b/2 > 0$. But p' is available to the q-player, denoted now as strategy q', and from the foregoing observation, for all $p E_{r_{0.5}}(p, q') \leq -b/2 < 0$. This contradicts T.2.3. □

Under the same Condition $\mathcal{A}$, next, we show that there exist solutions to the E_{r_w}-game within the metric completions of ρ_P and ρ_Q.

(Condition $\mathcal{A}$ is assumed for the complete spaces.) Not only does the game have a value, but minimax strategies exist. The central idea in this theorem is an application of a result about the nonempty intersection of closed sets in a complete metric space, due to Kuratowski (1968), using the closure of the sets $I^{\varepsilon}(p)$. The next lemma, attributed to Kuratowski, is the metrical analog to compactness for the familiar and elementary result that, in a compact space, if a family of closed sets has the finite intersection property, then the family has a nonempty intersection.

Following Kuratowski, denote by $\alpha(x)$ the greatest lower bound of numbers ε such that set x can be decomposed into a finite union of sets of diameter $< \varepsilon$.

Lemma 2.5 (Kuratowski, 1968, vol. 1, p. 412, the Corollary). In a complete metric space, let $\{F_\iota\}$ be a family (of arbitrary cardinality) of closed sets with the finite intersection property. If there are sets F_ι with arbitrarily small $\alpha(F_\iota)$, then the entire family has nonempty intersection. □

Consider the topology on (the equivalence classes of) P (or on Q) induced by ρ_P (or ρ_Q), respectively. The product topology on $P \times Q$ can be metrized by (among others):

$$\rho_{P\times Q}[(p_1, q_1), (p_2, q_2)] = \max\{\rho_P(p_1, p_2), \rho_Q(q_1, q_2)\}.$$

Take the metric completions ρ_P*, ρ_Q*, and $\rho_{P\times Q}*$ obtained by embedding the sets P, Q, and $P \times Q$ in the space of bounded, continuous (real-valued) functions (on P, Q, and $P \times Q$) using the supremum metric. (See Dugundji, 1968, p. 304). These completions are related by the next lemma.

Lemma 2.6. The space $(P \times Q)^*$ is (identically) isometric with the product space $P^* \times Q^*$, where $\rho_{P\times Q}*$ is the common metric. □

Proof. By corollary 5.3, p. 303, of Dugundji (1968).

Thus, we may identify limits from the space $(P \times Q)^*$ by taking limits from each player's space of strategies. We use this metric completion to produce (extended) strategies p^* and q^* that solve the E_{r_w}-game.

Theorem 2.4.

- **i.** Provided that Condition $\mathcal{A}$ obtains, $(\exists q^* \in Q^*)$ $(\forall p \in P^*)$ $E_{rw}(p, q^*) \leq V_w$.
- **ii.** Likewise, provided Condition $\mathcal{A}$ obtains with the players' roles reversed, $(\exists p^* \in P^*)(\forall q \in Q^*)E_{rw}(p^*, q) \geq V_w$.
- **iii.** So, if Condition $\mathcal{A}$ obtains for both players, $E_{rw}(p^*, q^*) = V_w$, and the strategy pair (p^*, q^*) solves the E_{rw}-game.

Proof. Regarding (i), we apply Kuratowksi's Lemma 2.5 to the *closed* sets $\{\mathbf{I}^{\varepsilon}(p): p \in P^*, \varepsilon > 0\}$, where closure is with respect to the complete space P^*. However, in order to do this, first we state a property concerning the extension of expectations from $(P \times Q)$ to $(P \times Q)^*$. The following two lemmas are easy to prove, the second being an immediate consequence of the first. (Also, they appear in Kretkowski and Telgárski, 1983.)

Lemma 2.7. E_{rw} has a unique, uniformly continuous extension form $(P \times Q)$ to $(P \times Q)^*$. □

Lemma 2.8.

$$\sup_P \inf_Q E_{rw}(p, q) = \sup_{P^*} \inf_{Q^*} E_{rw}(p, q)$$

and

$$\inf_Q \sup_P E_{rw}(p, q) = \inf_{Q^*} \sup_{P^*} E_{rw}(p, q). \quad \Box$$

Next, we duplicate Lemmas 2.2 and 2.3 for the space of metric completions P^* and Q^*. The proofs of Lemmas 2.9 and 2.10 follow exactly those of 2.2 and 2.3, respectively, and are omitted.

Lemma 2.9. Let $\sup_{P^*}\inf_{Q^*}E_{rw}(p, q) = a$. Then there is a finitely additive measure, μ, on the space of Q^*-strategies with the property that, $\forall p \in P^*, \int_{Q^*} E_{rw}(p, q)\, d\mu \leq a$. □

Based on Lemma 2.9, define the function $\vartheta^*(p) = \int_{Q^*} E_{rw}(p, q)\, d\mu$ and denote by $\mathbf{I}^{\varepsilon}(p)$ the closed set $\mathbf{I}^{\varepsilon}(p) = \{q \in Q^*: |E_{rw}(p, q) - \vartheta(p)| \leq \varepsilon\}$.

Lemma 2.10. The family $\{\mathbf{I}^{\varepsilon}(p): p \in P^*, \varepsilon > 0\}$ has the finite intersection property. □

Then part (**i**) of the theorem follows from Lemma 2.5, since Condition $\mathcal{A}$ assures the existence of closed sets $\mathbf{I}^{\varepsilon}(p)$ with arbitrarily small "α" (in Kuratowski's notation).

Part (**ii**) of the theorem is demonstrated by reversing the players' roles and part (**iii**) is an immediate consequence of (**i**) and (**ii**). □

II.2. *Non-convexity of the* E_{rw}*-Games, as a Function of* w

Our purpose in this subsection is to indicate, by example, that the minimax solution to an E_{rw}-game may fail to be a convex combination of the extreme solutions (where $w = 0$ and $w = 1$).

Example 2.1. Consider a modification of Wald's game where (as in "Pick the Bigger Integer") $f(s, t) = -1$ if $t > s$, $f(s, t) = 0$ if $s = t$, and $f(s, t) = 1$ if $2 \le t$ and $s > t$, but (unlike Wald's game) $f(s, t) = -0.5$ for $1 = t < s$. The row corresponding to the pure strategy $(t = 1)$ is altered.

It is straightforward to show that the game has values: $V_0 = -1$, $V_1 = -0.2$, and $V_{0.5} = -1/3$, for the parameter settings $w = 0$, $w = 1$, and $w = 0.5$, respectively. Let p_d *and* q_d denote any diffuse (p.f.a.) mixed strategies on the integers. Then the minimax strategies for the games are:

$$p_0^* \text{ is arbitrary (as } q_0^* \text{ is an "equalizer" strategy)}, \quad q_0^* = q_d$$

for $w = 0$;

$$p_1^* = 0.4p_d + 0.6\,(s = 1), \quad q_1^* = 0.8q_d + 0.2\,(t = 1)$$

for $w = 1$;

$$p_{0.5}^* = (2/3)p_d + (1/3)\,(s = 1), \quad q_{0.5}^* = (1/3)q_d + (2/3)\,(t = 1)$$

for $w = 0.5$.

Note that $V_{0.5} \neq 0.5V_0 + 0.5V_1$. That is, the value of the game for $w = 0.5$ is not the equal mixture of the values for the two extreme games. Though $V_{0.5} = (1/6)V_0 + (5/6)V_1$ and $q_{0.5}^* = (1/6)q_0^* + (5/6)q_1^*$, $p_{0.5}^*$ is not a similar mixture of p_0^* and p_1^*, for any p_0^*. The strategies for the mixed game are not a mixture of the strategies for the extreme games. In short, our proposal to use joint distributions which are the convex combination of two extreme distributions ($w = 0$ and $w = 1$), the E_{rw}-game, results in a non-convexity of minimax values and minimax strategies with respect to the parameter w.

Condition $\mathcal{A}$ applies (and in either order, provided $w > 0$) to the strategy spaces for this game. For example, with $w = 0.5$, $p_{0.5}^*$ equalizes on the two strategies used to define $q^*_{0.5}$: the diffuse q_d and the pure $(t = 1)$. Thus, each mixed strategy $xq_d + (1 - x)\ (t = 1)$ (for $0 \le x \le 1$) is a "good" response to $p^*_{0.5}$. Any other "good" response to $p^*_{0.5}$ is ρ_Q-close to one of these mixtures. Moreover, these mixtures are contained within n closed ρ_Q-balls (of radius $1/n$), where each closed ball is centered at the strategy $q_i = [i/n]q_d + [(n - i)/n]\ (t = 1)$ (for $1 \le i \le n$). To see that $\mathcal{A}$ is satisfied, choose $k = 3$ and, given $\varepsilon > 0$, let $n \ge 1/\varepsilon$. Thus, if $q \in R^{2\varepsilon}(p^*_{0.5})$ then $q \in \cup_i B\rho_Q(q_i, 3\varepsilon)$.

III. SOME COMPARISONS WITH OTHER WORK ON FINITELY ADDITIVE MIXED STRATEGIES

Investigation of finitely additive strategies in zero-sum, two-person games dates (at least) from Samuel Karlin's (1950) essay. Karlin, in turn, responds to Ville's (1938) game without a value, which uses a bounded, discontinuous payoff function on the unit square. By a separating hyperplane argument, Karlin shows that Ville's game does have a value when the players use finitely additive strategies and these are composed into a joint distribution *with the order of expectations fixed*. That is, Karlin's expectation for a (bounded) payoff function f on $[0, 1]^2$, using the finitely additive strategy pair (p, q), is based on the analysis of e.g., $E_p[E_q f(x, y)]$. Also, Karlin's proof relies on an assumption that the space of pure strategies available to a player is compact.

Heath and Sudderth (1972) extend Karlin's result to all two-person games with bounded payoff functions $f(s, t)$, again using a separating hyperplane argument. It is reported here as Theorem 2.1. [Their proof avoids Karlin's assumption that the space of pure strategies is compact.] But, like Karlin's work, their joint distributions are based on a fixed order of expectations. In terms of our proposal to use the family:

$$E_{r_w} = wE_q[E_p f(s,t)] + (1-w)E_p[E_q f(s,t)],$$

Heath and Sudderth's theorem establishes the (von Neumann) minimax result only for the two extreme cases: $w = 0$ and $w = 1$.

E. B. Yanovskaya (1970) expresses dissatisfaction with the required asymmetry of these solutions. (J. Kindler, 1983, adopts Yanovskaya's approach for this reason.) Yanovskaya introduces a method for assigning values to a strategy pair (p, q), and here that value is denoted $E^Y(p, q)$, which we paraphrase as follows: When $E_q[E_p f(s, t)] = E_p[E_q f(s, t)]$, then $E^Y(p, q)$ is defined by integration in either order. However, when $E_q[E_p f(s, t)] \neq E_p[E_q f(s, t)]$, then $E^Y(p, q)$ is stipulated to be some (real) quantity a. With this method, Yanovskaya shows that the minimax theorem (as in Theorem 2.3) obtains for a nonempty set of a values: either the set is a closed interval $[a_1, a_2]$, or it is a single value $[a]$. The special value $a^* = [a_1 + a_2]/2$ then is uniquely determined by an additional appeal to three invariance/symmetry conditions. In short, with Yanovskaya's proposal, all symmetric (bounded) zero-sum (two-person) games are fair, i.e., $a^* = 0$.

We are completely sympathetic with the objection that fixing the order of integration introduces undesirable asymmetries, as in Heath and Sudderth's solutions to Wald's game. However, we find Yanovskaya's proposal unsatisfactory, for the following reason. The $E^Y(p, q)$-numbers do not satisfy (finitely additive) expected utility theory. (The cogency of the three invariance conditions for choosing the midpoint of the $[a_1, a_2]$-interval is a different matter altogether.)

For instance, in connection with Wald's game (for which $a_1 = -1$, $a_2 = 1$ and thus $a^* = 0$), consider the three strategy pairs (p_1, q_d), (p_d, q_d), and (p_x, q_d), with $(0 < x < 1)$, where: p_d and q_d are diffuse, p_1 is the pure strategy $\{s = 1\}$, and $p_x = xp_1 + (1 - x)p_d$. Then $E^Y(p_1, q_d) = -1$, but $E^Y(p_d, q_d) = E^Y(p_x, q_d) = a^* = 0$. However, p_x is the simple mixture of two strategies p_1 and p_d; hence, according to (even finitely additive) expected utility theory, the values ought to satisfy: $E(p_x, q_d) = xE(p_1, q_d) + (1 - x)E(p_d, q_d)$. We understand this expectation feature of expected utility theory to provide the justification (in fact, von Neumann's justification) for assigning values to (countably additive) mixed strategies based on the values of pure strategies. It seems clear to us that in extending the value structure to include finitely additive mixed strategies, the simple expectation property (described above) is to be respected. Therefore we do not accept Yanovskaya's method for solving finitely additive games.

When discussing how to prove minimax theorems like Theorem 2.3 for infinite games (under the assumption that all strategies are countably additive), Karlin offers this advice.

> In the theory of infinite games, the truth of [the minimax theorem] is a deep question, requiring some kind of assumption of continuity [for joint expectations] and the restriction that at least one of the spaces [P and Q] is compact space in some suitable sense. (Karlin, 1959, p. 23)

We were unsuccessful in finding a way to duplicate the proofs given in section 2.1 (based on metric completions) using compactification of the spaces P, Q, and $P \times Q$. For one example, using Stone-compactification in Wald's game leads to a failure of the counterpart to Lemma 2.6 (see Glicksberg's theorem, 1959). For a second example, using the topology of "pointwise" convergence with Wald's game invalidates the counterpart to Lemma 2.7. Instead of compactification, we settled on Condition $\mathcal{A}$ as a useful alternative.

Last, it is straightforward to show that, provided Condition $\mathcal{A}$ obtains (or if $\mathcal{A}$ obtains with P and Q spaces reversed), the set of minimax strategies for q-player (or for p-player) is precompact with respect to ρ_Q (or ρ_P). This is in contrast with Fan's (1953) T.3(ii), minimax theorem. Fan's result requires (for our case) the assumption that E_{rw} be an almost periodic function on the product set $P \times Q$. "Almost periodicity" is both necessary and sufficient to make *all* of $P \times Q$ into a precompact, uniform space. Also, it is a symmetric condition on the product: right almost periodicity is equivalent to left almost periodicity. Neither of these feature is a consequence of Condition $\mathcal{A}$.

IV. SOME CONNECTIONS BETWEEN "IMPROPER" PRIORS AND F.A. MINIMAX STRATEGIES

Wald (1950) uses game theory to model nonsequential, statistical decisions, roughly, as follows: Player 1 is Nature. A pure strategy is the determination of a parameter $\theta \in \Theta$. Nature's mixed strategies are ("prior") distributions $p(\theta)$. The Statistician is Player 2. The Statistician observes an r.v. X, whose distribution $F_\theta(x)$ is given as a function of the parameter θ. (That is, the statistical model fixes the "likelihood" function for the game.) The Statistician has options, terminal decisions, $d^t \in D^t$, where a pure strategy d (a nonrandomized decision function) for the Statistician is a function from X to D^t. Payoffs to Nature ("losses" to the Statistician) are indicated by non-negative real numbers, $L(\theta, d^t) \geq 0$. Also, Wald's theory assumes losses are in "regret" form, i.e., $\forall \theta \, \exists d^t L(\theta, d^t) = 0$.

In a typical statistical decision problem each side has infinitely many pure strategies. Wald's treatment of statistical games imposes (asymmetric) mathematical conditions on the strategy sets for the two players. These conditions prove sufficient to insure that the game has a value. But the asymmetry leads to existence of a (mixed) minimax solution for the Statistician only: only Player 2 is assured a solution, using countably additive mixed strategies.

Example 4.1. *Point Estimation of a normal mean parameter* (known variance) *with squared-error loss.* (Strictly, this problem is not treated by Wald's theory, since squared-error loss is not bounded. That mathematical detail is irrelevant to the point illustrated here, however.)

"Point estimation" is the variety of problem where, given a potential observation x, the Statistician must propose a value for the (real-valued) parameter. The Statistician's pure strategies are of the form $d{:}X \to \Theta$. Squared-error loss (to the Statistician) is the payoff $(d - \theta)^2$, understood as Nature's gain.

Let $F_\theta(x)$ be the normal distribution with mean θ and unit variance $X \sim N(\theta, 1)$. Under squared-error loss the game is determined, with value $V = +1$. The Statistician's minimax strategy, a pure strategy, is the intuitive rule $d^*(x) = x$ – posit the observed value. However, no σ-additive mixed strategy for Nature has this large a maximin value. [Note: $\forall\theta\ E_\theta\,(d^* - \theta)^2 = 1$; so , d^* equalizes "risk."] Of course, Nature can approximate the maximin value +1 using countably additive mixed strategies. Specifically, if Nature chooses the mixed strategy $\pi_n(\theta) \sim N(0, n)$, then

$$\inf_D t E_{\pi_n}(d-q)^2 = n/(n+1) < 1 = V.$$

Thus, against π_n the Statistician can improve on d^* only by the amount $1/(n + 1)$.

Consider the sequence π_n of mixed (prior probability) strategies that approximate the value of the game for Nature:

$$\pi_n(\theta) \sim N(0, n), \quad n = 1, \ldots$$

The π_n strategies converge (weak-star) to a uniform measure on Θ: an "improper prior" on Θ, represented by (σ-finite) Lebesgue measure. The sequence π_n of $N(0, n)$ probabilities has subsequences converging (weak-star) to diffuse f.a. probabilities p^*: distributions that assign zero probability to each unit interval, $\forall k p^*(k \le \theta \le k + 1) = 0$. Moreover, $\inf_D t\ E_p{}^*(d - q)^2 = 1$. So, (p^*, d^*) solves this estimation game. Nature needs to play a (diffuse) finitely additive, mixed strategy to achieve its maximin value for the game.

V. CONCLUSIONS AND OPEN QUESTIONS

First we address a question about what might be intended by a f.a. mixed strategy. We know of five philosophical considerations that lead some to the use of f.a., rather than σ-additive probabilities. In increasing importance for our discussion here, they are:

1. Measurability precludes non-trivial, σ-additive probabilities on uncountable sets (Ulam, 1930). As is well known, a σ-additive probability can be extended to the power set provided the extension may be only finitely additive (as follows from the Hahn-Banach theorem). DeFinetti (1972, p. 201) and Dubins and Savage (1976, p. iii) have voiced this theme in justification of f.a. probability. In rebuttal, we point to the important result of Solovay (1970) which shows that every subset of the real line may be Lebesgue measurable if the Axiom of Choice is weakened and a suitable large-cardinal is introduced. There is more than one way to solve the measurability question.
2. Limits of relative frequencies (over infinite sets of possible outcomes) are not generally σ-additive measures. In rebuttal to this familiar observation there is the position (often voiced by "Bayesian" statisticians) that probability is not a limit of relative frequencies – the limiting frequency interpretation is not a useful one.
3. Some, important decision theoretic treatments of personal probability do not require more than finite additivity. For example, deFinetti's (1974) theory of coherent previsions and Savage's (1954) normative theory of preference allow for (merely) finitely additive personal probabilities, based on principles of rational choice.
4. Familiar (classical) statistical techniques often have Bayesian "models" that use diffuse finitely additive "prior" probabilities. These take the form of "improper" priors, e.g., in Jeffreys' (1971) theory.
5. Wald's treatment of statistical games often leads to maximin strategies for Nature which are purely f.a., as illustrated by Example 4.1 (above). The same example suggests that diffuse maximin strategies may be approximated by countably additive mixed strategies, as the π_n approximate p^*. Recall, however, Wald's game (Example 1.1), "Pick the Bigger Integer," and our fair diffuse minimax (and

maximin) solutions to the fully symmetric version, $E_{r0.5}$. No countably additive strategy is a good approximation to these. Nonetheless, the infinite game may be approximated by a sequence of trivial, symmetric finite games – "Pick the Biggest Integer $\leq k$." Obviously, each of these trivial, fair games is solved with a pure, point-mass strategy, $\{k\}$. But (ignoring the difference between choosing k for certain and choosing k with probability 1), each (weak-star) limit of this sequence of pure-strategies is a diffuse probability, corresponding to our proposal for f.a. solutions to the infinite game. Thus, the f.a. solutions to the infinite game "Pick the Bigger Integer" are approximated by θ-additive solutions to finite approximating games, rather than being approximated by σ-additive strategies within the infinite game. This observation leads us to our first open question:

Issue 1. How (and when) can f.a. minimax strategies be approximated by σ-additive ones? When is the approximation by a sequence of σ-additive strategies within the infinite game (as is possible in Example 4.1, but not in Example 1.1) and when is the approximation by a sequence of bounded games (as is possible in both examples)?

We conclude with several questions abut the adequacy of our mathematical approach to solving games using f.a. probabilities.

Issue 2. How generous is Condition $\mathcal{A}$? We have not indicated (because we do not know) when an infinite game satisfies $\mathcal{A}$. It may be worthwhile to investigate this question, even if all strategies are required to be σ-additive, as in traditional game theory. We say this because the minimax theorem, Theorem 2.3, obtains for countably additive game theory. Lemmas 2.2–2.4 apply when P and Q are sets of σ-additive probabilities. (Specifically, the f.a. measure μ of Lemma 2.2. appears there as a computational device for defining the integral $\vartheta(p)$ and the set $I^{\varepsilon}(p)$.) Of course, Condition $\mathcal{A}$ does not obtain in Wald's game, Example 1.1, when strategies are σ-additive; but it does obtain when strategies are f.a. probabilities. We would like to understand the circumstances that make $\mathcal{A}$ hold.

Issue 3. What are the mathematical entities introduced in the metric completion of Theorem 2.4? We mean to ask both: (i) When are the spaces $P \times Q$ metrically complete (for $\rho_{P\times Q}$)? and (ii) Are the minimax "strategies" formed by the completion of $\rho_{P \times Q}$ f.a. probabilities over some interesting algebra of events?

We hope to address these topics in our future work.

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